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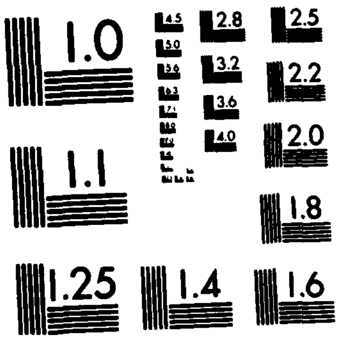

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Eigenvector Matrices of Symmetric Tridiagonals

B.N. Parlett[†] *and W-D Wu*[‡]

ABSTRACT

A simple test is given for determining whether a given matrix is the eigenvector matrix of an (unknown) unreduced symmetric tridiagonal matrix. A list of known necessary conditions is also provided. A lower bound on the separation between eigenvalues of tridiagonals follows from our Theorem 3.

This paper is dedicated to Professor F.L.Bauer on the occasion of his 65th birthday.

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1. INTRODUCTION.

In a sequence of papers during the 1960's Slepian, Landau, and Pollack investigated functions that are nearly time limited ($supp\ f \in [-T,T]$) and nearly band limited ($supp\ \hat{f} \in [-\Omega,\Omega]$), see [S.,1983]. Among other things they discovered that the integral operator K on $L^2[-T,T]$ given by

$$(Kf)(t) := \int_{-T}^{T} \frac{\sin\Omega(t-s)}{(t-s)} f(s)\, ds$$

commutes with a second order differential operator D given by

$$(Df)(x) := \left[(T^2-x^2)f'(x)\right]' - \Omega^2 x^2 f(x)$$

and zero boundary conditions. Here f' denotes the derivative of f. Moreover D has a simple (point) spectrum. Consequently the eigenfunctions of K are the eigenfunctions of D. The authors managed to generalize this result from x and t in R to x and t in R^N. A decade later Slepian demonstrated the discrete

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analogue [S., 1978]: K becomes a symmetric Toeplitz matrix with (i,j) element

$$\frac{\sin 2\pi\omega(i-j)}{2\pi(i-j)}, \quad -n \le i,j \le n$$

and D becomes a symmetric tridiagonal matrix. This is a rare example of a Toeplitz matrix which can be expressed as a polynomial in a non-Toeplitz matrix.

Recently Grunbaum has been extending these results in several directions. For example, in [G., 1981] he describes all symmetric Toeplitz matrices which admit some unreduced tridiagonal matrix in their commutators. It turns out that each such matrix is determined (essentially) by just three of its elements: (1,2), (1,3), and (1,4). In [G., 1982] he exhibits Hankel matrices (including the Hilbert matrix) that commute with unreduced tridiagonals.

It is well known [M. & M., p.77] that if A commutes with M and A has simple spectrum, then M is a polynmial in A and they share the same eigenvectors. What permits a symmetric matrix A to have a tridiagonal in its commutator is that its set of eigenvectors is very special. The matrix of eigenvectors may be taken as orthogonal and this communication characterizes the eigenvector matrix of a symmetric, unreduced tridiagonal matrix by means of a simple test. It turns out that the first two rows (or the last two rows) determine all the others, because they fix the eigenvalues.

After introducing notation and reducing the general problem, we give a list of known properties of these eigenvectors. Our characterization is given in Section 5. A bound on the number of zero elements in a row is given in Section 6 along with a result connecting eigenvalue separation and the top row of the normalized eigenvector matrix.

2. NOTATION AND DEFINITIONS

Capital roman letters denote matrices, with A and T reserved for real symmetric matrices. The elements of B are written B(i,j). Lower case letters x, y,

... denote column vectors, lower case Greek letters α, β, ..., denote scalars, but i, j, k, l, m, n are reserved for indices. G^t denotes the transpose of G.

A matrix C is <u>tridiagonal</u> if $C(i,j) = 0$ whenever $|i-j| > 1$. A tridiagonal matrix C is <u>unreduced</u> if $C(i, i+1) \neq 0$ and $C(i+1, i) \neq 0$ for all relevant indices i. The <u>spectrum</u> of C is the set of eigenvalues of C, ignoring multiplicity.

For brevity let $\mathbf{UST}(n)$ be the set of $n \times n$ real unreduced, symmetric, tridiagonal matrices. Let $\mathbf{UST}_+(n)$ denote the subset of **UST** in which the $(i,i+1)$ elements are positive.

Eigenvectors are defined only to within a scalar factor. However, the eigenvecrtors of a real symmetric matrix may be chosen to be orthonormal, and this is a natural convention that we too will follow. When A has simple spectrum we may speak of **the** eigenvector matrix G and write

$$A = G \Lambda G^t$$

where Λ is diagonal and G is <u>orthogonal</u>. For a given ordering of the eigenvalues in Λ the matrix $G = (g_1, \ldots, g_n)$ is <u>unique</u> (up to replacement of g_i by $-g_i$). If $T = T_{1,n} \in \mathbf{UST}$ we define its submatrix $T_{j,n}$ by

$$T_{j,n} := \begin{bmatrix} \alpha_j & \beta_j & & & \\ \beta_j & \alpha_{j+1} & \beta_{j+1} & & \\ & \beta_{j+1} & \ddots & \ddots & \\ & & \ddots & & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix}$$

The characteristic polynomial of $T_{j,n}$ is

$$\chi_{j,n}(\zeta) := \det[\zeta I - T_{j,n}]$$

$\chi_{1,n}$ is abbreviated by χ or χ_n. The spectral factorization of $T = T_{1,n} \in \mathbf{UST}(n)$ is written

$$T = S \Theta S^t$$

where $\Theta := diag(\vartheta_1, \ldots, \vartheta_n)$ with eigenvalue ordering

$$\vartheta_1 < \vartheta_2 < \cdots < \vartheta_n,$$

and $S := (s_1, \ldots, s_n)$ is the orthogonal eigenvector matrix. Thus

$Ts_i = s_i \vartheta_i$, $i = 1, \ldots, n$, and $s_i^t s_i = 1$.

3. PROBLEM REDUCTION.

The identity matrix I is tridiagonal and consequently any orthogonal matrix is an eigenvector matrix of some tridiagonal. This case shows the importance of insisting on a simple spectrum for T.

Consider next tridiagonal matrices T which are reduced $\left[T(j,j+1) = 0 \text{ for } some\ j\right]$ and yet have simple spectrum. Such T's are the direct sum of two or more unreduced tridiagonals, say $T = diag(T_1, T_2, T_3)$, with $T_i \in$ **UST**. Its eigenvector matrix is the corresponding direct sum of the eigenvector matrices of T_1, T_2, and T_3 provided that the eigenvalues of T_i are kept together in the ordering of the eigenvalues of T. However, no matter what the ordering of the eigenvalues, T's eigenvector matrix is not fully indecomposable. We follow the terminology of [M. & M., 1964] here; some authors would say that T's eigenvector matrix is reducible.

The procedure for deciding whether a given orthogonal Q is the eigenvector matrix of some tridiagonal with simple spectrum is to column-permute Q into a direct sum of its fully indecomposable parts and then to determine whether each part is the eigenvector matrix of some T_i in **UST**.

4. KNOWN PROPERTIES.

FACT 1. If $T \in$ **UST**(n) then T has n distinct real eigenvalues.

Proof. The minor of the $(1,n)$ element of $T-\zeta I$ is $\prod_{i=1}^{n-1} T(i+,i)$ which vanishes for no value of ζ. So rank $(T-\zeta I) \geq n-1$. The eigenspace of each eigenvalue has dimension 1. The result follows from the symmetry of T. $\blacksquare$

WARNING: The eigenvalues of T may be very close indeed, even when the off diagonal elements are not small relative to the eigenvalue spread, but see

Theorem 3 in Section 6.

FACT 2. If $T \in \mathbf{UST}(n)$ then the eigenvalues of $T_{1,n-1}$ separate those of $T_{1,n}$. The eigenvalues of $T_{2,n}$ separate those of $T_{1,n}$. ▪.

Proof. See [P., 1980, Chap. 10].

The next result, which uses the notation established in Section 2, was extended by Gantmacher and Krein [G. & K., 1950] to the class of oscillation matrices. Karlin has extended their results to include differential operators [K., 19].

Recall that the eigenvalues of $T_{1,n}$ are ordered by

$$\vartheta_1 < \vartheta_2 < \cdots < \vartheta_n.$$

We sketch a proof because it is short.

FACT 3. If $T \in \mathbf{UST}_+(n)$ then the number of sign changes between consecutive components of ϑ_i 's eigenvector s_i is n-i, for i = 1, ..., n.

Proof. By the 3 term recurrence governing the $\chi_{1,i}$ one can verify that s_i is proportional to

$$[1, \chi_1(\vartheta_i)/\beta_1, \chi_2(\vartheta_i)/\beta_1\beta_2, \ldots, \chi_{n-1}(\vartheta_i)/\beta_1 \cdots \beta_{n-1}]^t$$

and each $\beta_i > 0$. The polynomials $\{1, \chi_1, \ldots, \chi_{n-1}\}$ form a Strum sequence. (See [W., 1965, p. 300] but observe that $\det[T - \zeta I] = (-1)^n \chi_n(\zeta)$). It follows that the number of sign differences in $\{1, \chi_1(\zeta), \ldots, \chi_{n-1}(\zeta)\}$ equals the number of eigenvalues of $T_{1,n}$ greater than ζ, not less than ζ. The result follows from Fact 2.

The following more detailed results can be found in [P., 1980, Chap. 7].

FACT 4. (C.C. Paige). If $T \in \mathbf{UST}(n)$ then

a) $s_i(1)^2 \chi'(\vartheta_i) = \chi_{2,n}(\vartheta_i)$,

b) $s_i(n)^2 \chi'(\vartheta_i) = \chi_{n-1}(\vartheta_i)$,

c) $s_i(1) s_i(n) \chi'(\vartheta_i) = \beta_1 \beta_2 \cdots \beta_{n-1}$.

COROLLARY. $s_i(1) \neq 0, s_i(n) \neq 0,$, $i = 1, \ldots, n$.

Proof. By Fact 2, $\chi_{2,n}(\vartheta_i) \neq 0$, $\chi_{n-1}(\vartheta_i) \neq 0$. By Fact 1, $\chi'(\vartheta_i) \neq 0$. ▪.

The corollary is a well known necessary condition for an orthogonal matrix to be an eigenvector matrix for some $T \in \mathbf{UST}$. In the next section we generalize this result to the other rows.

If $T \in \mathbf{UST}(n)$ then there is no loss of generality in assuming that $T \in \mathbf{UST}_+(n)$. More precisely, if $T \in \mathbf{UST}$ then there is a diagonal matrix Δ with $\Delta(i,i) = \pm 1$ such that $\Delta T \Delta \in \mathbf{UST}_+$. If S is the eigenvector matrix of T, then ΔS is the eigenvector matrix of $\Delta T \Delta$. Moreover, we may take $s_i(1) > 0$, $i = 1, \ldots, n$.

FACT 5. (Uniqueness of Reduction.) If $Q^t A Q = T \in \mathbf{UST}_+(n)$, and Q is orthogonal, then Q and T are determined by A and q_1 or by A and q_n.

Proof. See [P., 1980, Chap. 7].

COROLLARY. Let $S \Theta S^t = T \in \mathbf{UST}(n)$ be the spectral factorization of T. Both T and S are determined by Θ and $e_1^t S$ or by Θ and $e_n^t S$. Our last fact makes explicit the dependence of Q on q_1 and A in Fact 5. The formula goes back at least to C. Lanczos.

FACT 6. Let $T = Q^t A Q \in \mathbf{UST}_+(n)$ with Q orthogonal.

$$q_{j+1} = \chi_j(A) q_1 / (\beta_1 \cdots \beta_j), \text{ for } j = 1, \ldots, n-1.$$

Proof. See [P., 1980, p. 116].

5. CHARACTERIZATION OF THE EIGENVECTOR MATRIX

There is a whole family of tridiagonals which share a common eigenvector matrix.

LEMMA 1. If $T \in \mathbf{UST}_+$ then $\lambda T - \sigma I \in \mathbf{UST}_+$ for all σ and all $\lambda > 0$. Moreover T and $\lambda T - \sigma I$ have the same eigenvector matrix.

Proof. The first asertion is trivial. The second uses the orthogonality of the eigenvector matrix S; $T = S \Theta S^t$ implies $\lambda T - \sigma I = S(\lambda \Theta - \sigma I) S^t$. ▪

This trivial lemma is the key to a simple characterization of S. It shows that in the search for a T there is no loss in taking $\alpha_1 = T(1,1) = 0$ and $\beta_1 = T(1,2) = 1$.

THEOREM 1. Let $G = (g_1, \ldots, g_n)$ be orthogonal. G^t is the eigenvector matrix of some $T \in \mathbf{UST}(n)$ if and only if the following conditions hold.

(1) $g_1(i) \neq 0, i = 1, \ldots, n$.

(2) The numbers ϑ_i are all distinct, where $\vartheta_i := g_2(i)/g_1(i), i = 1, \ldots, n$.

(3) $G^t \Theta G \in \mathbf{UST}$, where $\Theta = diag(\vartheta_1, \ldots, \vartheta_n)$.

Proof. Condition 1 was established as the Corollary of Fact 4.

Because G is orthogonal, $G^t \Theta G = T$ can be rewritten $\Theta G = GT$. Equating column 1 on each side yields

$$\Theta g_1 = g_1 \alpha_1 + g_2 \beta_1.$$

By Lemma 1 there is no loss in taking $\alpha_1 = 0, \beta_1 = 1$, to obtain $\vartheta_i g_1(i) = g_2(i), i = 1, 2, \ldots, n$. Thus Θ is determined by g_1 and g_2. By Fact 1 the ϑ_i must be distinct. This is condition 2.

If $G^t \Lambda G \in \mathbf{UST}(n)$ for some $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$, then, by the previous paragraph, $\Lambda g_1 = g_1 \alpha_1 + g_2 \beta_1$ for some α_1, β_1 and thus $\lambda_i = \alpha_1 + \vartheta_i \beta_1, i = 1, \ldots, n$. Consequently $G^t \Theta G = (G^t \Lambda G - \alpha_1 I)/\beta_1 \in \mathbf{UST}(n)$ as well. This establishes condition 3. ∎

Condition 3 is easily stated but will fail for most orthogonal matrices satisfying conditions 1 and 2. We can replace (3) with a sequence of necessary conditons. As soon as one fails, we know that G^t cannot be an eigenvector matrix.

DECISION PROCEDURE. Given distinct values $\vartheta_i := g_2(i)/g_1(i)$, i = 1, ..., n, and $\Theta = diag(\vartheta_1, \ldots, \vartheta_n)$ proceed as follows:

for $j = 2, \ldots, n-1$, do

form $\hat{g}_{j+1} := \Theta g_j - g_j \alpha_j - g_{j-1} \beta_{j-1}$,

where $\alpha_j := g_j^t \Theta g_j$, β_j [illegible]

If $g_{j+1} \neq \pm \hat{g}_{j+1} / \|\hat{g}_{j+1}\|$ then exit and report failure.

if no failure then G^t is an eigenvector matrix for the T defined by the α_i and β_i.

Proof. By Fact 5 (uniqueness of reduction) there is a unique orthogonal matrix $\overline{G}$ and a unique T_+ in **UST**$_+$ satisfying

$$T_+ = \overline{G}^t \Theta \overline{G},\ \bar{g}_1 = g_1.$$

All T in **UST** orthogonally similar to Θ are given by

$$T = \Delta T_+ \Delta = \Delta \overline{G}^t \Theta \overline{G} \Delta$$

with diagonal Δ and $\Delta(i,i) = \pm 1$. Hence $G^t \Theta G \in$ **UST** if and only if $G = \overline{G}\Delta$ for some Δ.

The algorithm in the procedure is simply the construction of the matrix $\overline{G}$ column by column together with the test $g_i = \pm \bar{g}_i$. ∎

There is a result dual to Theorem 1 which uses the last row instead of the first one. We state it without proof.

Theorem 1'. Let $G = (g_1, \ldots, g_n)$ be orthogonal. G^t is the eigenvector matrix of some $T \in$ **UST**(n) if and only if the following conditions hold.

1.' $g_n(i) \neq 0,\ i = 1, \ldots, n.$

2.' The numbers $\varphi_i (:= g_{n-1}(i)/g_n(i))$ must be distinct.

3.' $G^t \Phi G \in$ **UST**, where $\Phi = diag(\varphi_1, \ldots, \varphi_n)$.

Since different normalizations are involved, we will not have $\vartheta_i = \varphi_i$ in general. However they must be lineraly related,

$$\vartheta_i - \beta_{n-1}\varphi_i = \alpha_n,\ i = 1, \ldots, n.$$

In practice the decision procedure could work simultaneously from the top row and the bottow row towards the middle.

6. ZERO ELEMENTS.

THEOREM 2. If S is the eigenvector matrix of $T \in$ **UST**(n) then the number of zero elements in its j'th row is at most min (j-1, n-j).

Proof. Since $T = \Delta T \Delta$ for $T_+ \in$ **UST**$_+$ and $\Delta(i,i) = \pm 1$ there is no loss in considering $T \in$ **UST**$_+(n)$. Apply Fact 8 to the relation $T = S\Theta S^t$ to conclude that

$S^t e_{j+1}(\beta_1 \cdots \beta_j) = \chi_j(\Theta)S^t e_1$, i.e.,

$$s_i(j+1)(\beta_1 \cdots \beta_j) = \chi_j(\vartheta_i)s_i(1),\ i = 1, \ldots, n.$$

Now, χ_j is a polynomial of degree j and independent of ϑ_i. It can have at most j zeros and so $S(j+1,.)$ can have at most j zero elements. These occur when, and only when, an eigenvalue of T is also an eigenvalue of a submatrix $T_{1,j}$.

By Fact 5 (uniqueness of reduction) T and S are also determined by Θ and s_n. The analogous formula is

$$s_i(j+1)(\beta_{j+1} \cdots \beta_n) = \chi_{j+1,n}(\vartheta_i)s_i(n),\ i = 1, \ldots, n.$$

Since $\chi_{j+1,n}$ is of degree n-j-1, $S(j+1,.)$ can have at most n-j-1 zero elements and these occur when an eigenvalue of T is also an eigenvalue of $T_{j+1,n}$. ∎

Our last result concerns the separation of the ϑ's.

THEOREM 3. For $T \in \mathbf{UST}_+(n)$,

$$\prod_{\substack{i,j=1\\ i<j}}^{n} (\vartheta_j - \vartheta_i)/\beta_i = 1/\prod_{l=1}^{n} S(1,l) \geq n^{\frac{1}{2}n}$$

Proof. Define

$$B := \textit{diag}\,(1, \beta_1, \beta_1\beta_2, \ldots, \beta_1\beta_2 \cdot\cdot \beta_{n-1}),$$
$$\Sigma := \textit{diag}\,(S(1,1), S(1,2), \cdots, S(1,n)).$$

The displayed equation in the proof of Theorem 2 implies that

$$S = B^{-1}C\Sigma, \quad \text{where } C(i,j) := \chi_{i-1}(\vartheta_j).$$

By standard properties of determinants and Vandermondes

$$\det C = \det V = \prod_{\substack{i,j=1\\ i<j}}^{n} (\vartheta_j - \vartheta_i) > 0,$$

where $V(i,j) = \vartheta_j^{i-1}$. Since S is orthogonal,

$$\pm 1 = \det S = \det C \times \det\Sigma / \det B.$$

By our normalizations, including $S(1,l) > 0$, the right side is positive and so det S = +1. Finally

$$\det B = \beta_1^{n-1}\beta_2^{n-2} \cdots \beta_{n-1} = \prod_{\substack{i,j=1\\ i<j}}^{n} \beta_i.$$

$$\det\Sigma = \prod_{l=1}^{n} S(1,l) \le \left[\frac{1}{\sqrt{n}}\right]^n = n^{-\frac{1}{2}n}.$$

The last inequality comes from the generalized Arithmetic-Geometric mean relation. ▪

We quote without proof the analogous result using the last row of S.

THEOREM 4. For $T \in \mathbf{UST}_+(n)$,

$$\prod_{\substack{i,j=1\\ i<j}}^{n} (\vartheta_j - \vartheta_i)/\beta_{j-1} = 1/\prod_{l=1}^{n} |S(n,l)| \ge n^{\frac{1}{2}n}.$$

These results are informative. As $\beta_k \to 0$ so can one or more of the differences $\vartheta_{i+1} - \vartheta_i \to 0$ without violating the lower bound. Of more interest is the case of known fixed β's. The differences of ϑ's is bounded above by the spread $\vartheta_n - \vartheta_1$ which is itself bounded by $2\|T\|$. This yields a crude but explicit lower bound on $\min(\vartheta_{i+1} - \vartheta_i)$ in terms of the β's and $\| T \|$. Although eigenvalues of T's in $\mathbf{UST}(n)$ can be surprisingly close to each other, they cannot be arbitrarily close when the β's are given.

The relationships are even clearer in the special but important case of $\beta_i = 1$ for all i. For small values of n the presence of tiny $S(1,l)$ precludes close ϑ's but as n increases this effect weakens rapidly. Wilkinson's matrix W_{21}^{+} is an example of this phenomenon. See [W,p.308] for more information on this matrix.

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